March 29, 2000 WOCOMAL VARSITY MEET

ROUND I: Number theory

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. If N written in base 2 is 11000, then how do you write in base 2 the integer 1 smaller than N? (Just it, not as 11000-1)

2. The difference of the squares of two positive odd integers is always divisible by several numbers. What is the largest of them?

3. 1+2+3+4+5+6+7+8+9+10+11+12=78. Write all other ways which express 78 as the sum of consecutive positive integers.

ANSWERS

1. (1 pt) ______ two ______

2. (2 pts) __________

3. (3 pts) ____

Algonquin, Bancroft, Westborough

ROUND II: Algebra 1 - open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Determine the value of k: $(81)^2(27)^3 = 3^{6k}$

2. If P pencils cost C cents, how many pencils can be purchased for D dollars?

3. The values of a, b, and c are such that a-b = b-c = 3. Determine the value of $a^2 - 2b^2 + c^2$.

ANSWERS

1. (1 pt) ____________________

2. (2 pts) ____________________

3. (3 pts) ____________________

Hudson, St. John's, Tantasqua

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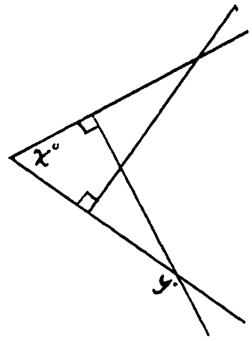
ROUND III: Geometry - open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM or ROUNDED TO THREE PLACES TO THE RIGHT OF THE DECIMAL POINT

1 . In the figure, express y in terms of x.



2. "Let your fingers do the walking" goes the Yellow Pages ad. I figured out the distance that one finger "walked" when I pressed 385-5395 on my phone, which has the following arrangement of keys:

1	2	3
4	5	6
7	8	9

If the rows and columns are 1 unit apart and I always took the shortest route, how far did my finger walk?

3. The largest possible regular octagon is inscribed in a 1 by 1 square. Find its area.

ANSWERS

1. (1 pt) $y =$ ____________________

2. (2 pts) ____________________

3. (3 pts) ____________________ sq. units

Bancroft, Burncoat, Mass. Academy

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ROUND IV: Logs, exponents, radicals

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Solve for y: $\sqrt[4]{\sqrt{x^{13}}} = x^y$

2. If $a^x = b$ and $b^y = a$, express y in terms of x.

3. Solve for x: $\left(\frac{x}{9}\right)^{\log 9} = \left(\frac{x}{11}\right)^{\log 11}$, $x > 0$

ANSWERS

1. (1 pt) ________________

2. (2 pts) $y =$ ________________

3. (3 pts) ________________

Clinton, St. John's, Shrewsbury

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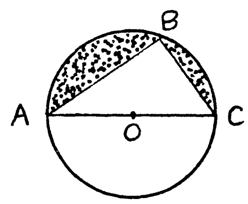
ROUND V: Trigonometry - open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM or AS DIRECTED IN THE PROBLEM

1. Simplify: $(1+\tan^2 x)(3-3\sin^2 x)$

2. Let $f(x) = 2\cos 2x + 3\sin 3x$. Find the period of f, in radians and in terms of π.

3. The radius of circle O is 4 and the shaded area equals the area of traingle ABC. To the nearest degree find the value for angle A which is less than 45^0.



ANSWERS

1. (1 pt) ______________

2. (2 pts) ______________

3. (3 pts) ______________

Auburn, Bancroft, Shrewsbury

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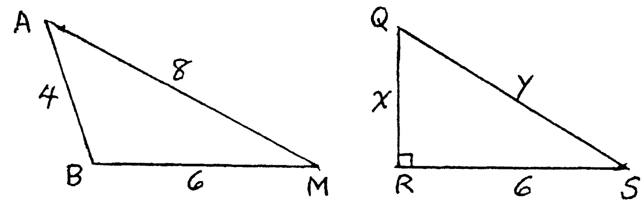
TEAM ROUND; Topics of previous rounds and open 2 POINTS EACH

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM except for number 5

1. Find the product of the LCM and the GCF of 220_3 and 33_4 in base 10.

2. Solve for A: $\dfrac{-2x-16}{x^2+x-2}=\dfrac{4}{x+2}+\dfrac{A}{x-1}$.

3. Find y if the area of triangle ABM is equal to the area of right triangle QRS.



4. If $x>1$ and $x=2+\sqrt{2+\sqrt{2+\sqrt{2+\dots\dots}}}$, find the exact value of x.

5. Points A and B are on one side of a straight river, 300 feet apart. Point C is on the opposite side and is situated so that angle CAB = 62^o and angle ABC = 54^o. Find the width of the river at C to the nearest foot.

6. What is the probability of choosing at random a regular polygon with less than 10 sides that will tesselate from all the possible regular polygons with less than 10 sides? Give your answer as a reduced fraction. Assume that each type polygon is equally likely.

7. There are 120 five digit numbers that can be formed using all 5 digits 1,2,3,4,and 5. Let these 120 numbers be arranged in increasing order. What is the 76th number in this order?

8. Graph on the number line: $||x-2|-2|\le 1$. You must supply relevant coordinates.

9. The first 27 numbers in the sequence 3, 33, 333, 3333, 33333, ... are added together. What is the digit in the thousands place in their sum?

Bancroft, Bromfield, Hudson, Quaboag, Tahanto, Worcester Academy

ROUND I: # thry

1. (1 pt) 10111_{two}
2. (2 pts) 8
3. (3 pts) 25 + 26 + 27 (= 78)
   18 + 19 + 20 + 21 (= 78)
   Need both (may omit "= 78")

ROUND II: alg 1

1. (1 pt) $\frac{17}{6}$ or $2\frac{5}{6}$ or $2.8\overline{3}$
2. (2 pts) $\frac{100PD}{C}$
3. (3 pts) 18

ROUND III: geom

1. (1 pt) $x + 90$
2. (2 pts) $3 + 2\sqrt{2} + \sqrt{5}$ or 8.064
3. (3 pts) $2\sqrt{2} - 2$ or 0.828 or $2(\sqrt{2} - 1)$

ROUND IV: logs exp rad

1. (1 pt) $\frac{13}{8}$ or $1\frac{5}{8}$ or 1.625
2. (2 pts) $y = \frac{1}{x}$ or x^{-1}
3. (3 pts) 99

ROUND V: trig

1. (1 pt) 3
2. (2 pts) 2π
3. (3 pts) $26°$

TEAM ROUND 2 pts each

1. 360
2. -6
3. $\sqrt{51}$ or 7.141
4. 4
5. 238 ft
6. $\frac{3}{7}$
7. 41,352
8. 

   may use [ ] for endpts
9. 0

March 29, 2000 — WOCOMAL VARSITY MEET — BRIEF SOLUTIONS

ROUND I

1.
$$\begin{array}{r} 11000 \\ -\ \ \ \ 1 \\ \hline 10111 \end{array} \qquad \begin{array}{r} 24 \\ -1 \\ \hline 23 \end{array}$$

2. The smallest possible difference of the squares of two pos. odd numbers is $9-1=8$, so the largest divisor is 8 or smaller. Write $\text{odd}^2 - \text{odd}^2$ as
$$(2m+1)^2 - (2n+1)^2$$
$$= 4m^2 + 4m + 1 - 4n^2 - 4n - 1$$
$$= 4(m^2 - n^2 + m - n)$$
$$= 4(m-n)(m+n+1)$$
Exactly one of these two factors is even, a mult. of 2, so the whole expression is a mult. of 8.

3. The average of an odd number of consecutive integers is the middle one and the ave. of an even number of such integers is the ave. of the middle two. For the sum of n consecutive integers to be 78, the average must be $\frac{78}{n}$.

n	need ave	possible?	
2	39	no	
3	26	yes	25+26+27
4	19.5	yes	18+19+20+21

$n = 5$ through 11 are not possible.
$n = 12$ gets the given way.
$n \geq 13$ makes the sum > 78

ROUND II

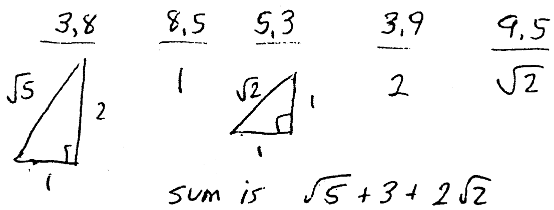
1. $81^2 \cdot 27^2 = (3^4)^2 \cdot (3^3)^3 = 3^8 \cdot 3^9 = 3^{17}$
$\therefore 17 = 6K$ and $K = \frac{17}{6}$

2. $\frac{P}{C} = \frac{x}{100D} \leftarrow$ cents, $x = \frac{100PD}{C}$

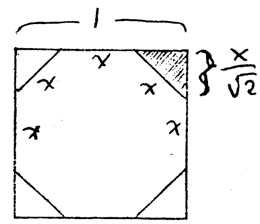
3. Get the desired expression in terms of b as
$$(b+3)^2 - 2b^2 + (b-3)^2$$
$$= b^2 + 6b + 9 - 2b^2 + b^2 - 6b + 9$$
$$= 18$$

ROUND III

1. An exterior of a $\triangle$ = the sum of the two remote interior angles.
$$y = x + 90$$

2.


sum is $\sqrt{5} + 3 + 2\sqrt{2}$

3.


One side of sq.
$$2\left(\frac{x}{\sqrt{2}}\right) + x = 1$$
$$x(\sqrt{2}+1) = 1$$
$$x = \frac{1}{1+\sqrt{2}}$$

Shaded $\triangle$ area $= \frac{1}{2}\left(\frac{x}{\sqrt{2}}\right)^2 = \frac{x^2}{4}$

Octagon area $= 1 - 4\left(\frac{x^2}{4}\right) = 1 - x^2$
$$= 1 - \frac{1}{(1+\sqrt{2})^2} \approx 0.828$$

ROUND IV

1. $\sqrt[4]{\sqrt{x^{13}}} = \left(\left(x^{13}\right)^{\frac{1}{2}}\right)^{\frac{1}{4}} = x^{\frac{13}{8}} = x^y$
$$y = \frac{13}{8}$$

2. $a^x = b \Rightarrow a = b^{\frac{1}{x}}$
Then $b^y = a = b^{\frac{1}{x}}$ and $y = \frac{1}{x}$

3. log each side
$$\log\left(\frac{x}{9}\right)^{\log 9} = \log\left(\frac{x}{11}\right)^{\log 11}$$
$$\log 9(\log x - \log 9) = \log 11(\log x - \log 11)$$
$$(\log 11)^2 - (\log 9)^2 = \log x(\log 11 - \log 9)$$
$$\log 11 + \log 9 = \log x$$
$$\log(11 \cdot 9) = \log x$$
$$x = 99$$

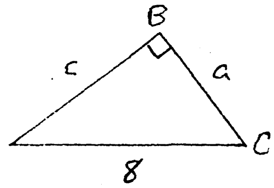
ROUND V

1. $(1+\tan^2 x)(3-\sin^2 x)$
$= (\sec^2 x)(3)(\cos^2 x) = 3$

2. $2\cos 2x$ has period $\frac{2\pi}{2} = \pi$
$3\sin 3x$ " " $\frac{2\pi}{3}$

period of sum is LCM, 2π

3. Area of rt $\triangle ABC$ is $\frac{1}{4}$ circle area $= 4\pi$ and also $= \frac{1}{2}ac$.



$\sin A = \frac{a}{8} \Rightarrow a = 8\sin A$

$\cos A = \frac{c}{8} \Rightarrow c = 8\cos A$

$\therefore\ 4\pi = \frac{1}{2}\cdot 8\sin A \cdot 8\cos A$

$\pi = 4\cdot 2\sin A\cos A$

$\frac{\pi}{4} = \sin 2A$

$2A = \sin^{-1}\frac{\pi}{4}$ and $A = \frac{1}{2}(51.7575\ldots)$
$= 26°$

TEAM ROUND

1. $220_3 = 2\cdot 9 + 2\cdot 3 = 24$
$33_4 = 3\cdot 4 + 3 = 15$.
Product of LCM and GCF
= product of two numbers
$= 24\cdot 15 = 360$

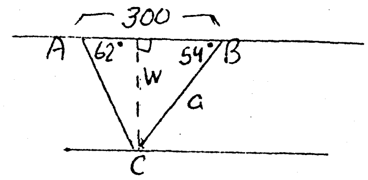
2. $\frac{-2x-16}{(x+2)(x-1)} = \frac{4(x-1)+A(x+2)}{(x+2)(x-1)}$

$\therefore\ -2x-16 = 4x-4+Ax+2A$
$-6x-12 = Ax+2A \Rightarrow A = -6$

3. Heron's formula on $\triangle ABM$:
$\sqrt{9\cdot 1\cdot 3\cdot 5} = 3\sqrt{15}$
Area of $\triangle QRS = \frac{1}{2}\cdot 6x = 3x$ } $x = \sqrt{15}$
$x^2+6^2 = y^2$ becomes $15+36 = y^2$
and $y = \sqrt{51}$

4. Change to $x-2 = \sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$
$x^2-4x+4 = 2+\sqrt{2+\sqrt{2+\cdots}}$
$= x$
$x^2-5x+4 = 0$
$(x-4)(x-1) = 0 \Rightarrow x = 4$

5. $\angle ACB = 64°$
Law of sines



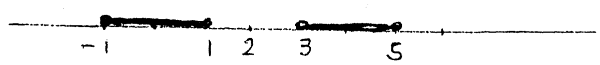
$\frac{a}{\sin 62°} = \frac{300}{\sin 64°}$ gets $a = 294.71\ldots$

Then in rt $\triangle$, $\sin 54° = \frac{w}{294.71}$
gets $w = 238$ ft

6. 7 possible polygons with 3, 4, 5, .., 9 sides. Of these those with 3, 4, and 6 sides tesselate. $\therefore\ \frac{3}{7}$

7. There are $\frac{120}{5} = 24$ beginning with each digit. $3\cdot 24 = 72$, so the 73rd one starts with 4. We want the 4th one starting with 4.
41235, 41253, 41325, <u>41352</u>

8. Think of $||x-2|-2| \le 1$ as $|a-2| \le 1$.
This means that a is 1 unit or less away from 2. $\therefore\ 1 \le a \le 3$.
Then $1 \le |x-2| \le 3$ so that x is between 1 and 3 units inclusive from 2.



9. $3\cdot 27 = 81$ so the units digit of the sum is 1. $3\cdot 26+8 = 86$, so the tens digit is 6. $3\cdot 25+8 = 83$, so the hundreds digit is 3. $3\cdot 24+8 = 80$, so the thousands digit is 0.